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ON THE THEORY OF ELASTIC NONHOMOGENEOUS MEDIA WITH A REGULAR STRUCTURE

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We give the theoretical description of regular elastic structures in an unbounded elastic medium with congruent (doubly-periodic) groups of arbitrary foreign inclusions. Within the limits of a group the elastic characteristics of the inclusions are distinct and their configurations are sufficiently arbitrary. We construct a model anisotropic medium which has the rigidity of the original structure. References on problems of the theory of elastic regular structures can be found in [1 - 3].

1. Formulation of the basic problem. We consider an elastic nonhomogeneous medium with a periodic structure. Let ω_1, ω_2 ($\mathrm{Im}\,\omega_1 = 0$, $\mathrm{Im}\,\omega_2/\omega_1 > 0$) be its fundamental periods. Inside the parallelogram of periods Π_{mn} ($m, n = 0, \pm 1, \pm \ldots$) a group of nonintersecting inclusions is contained. Each of the inclusions occupies a finite simply-connected domain D^j_{mn}, bounded by a simple, closed, sufficiently smooth contour L^j_{mn} ($j = 1, 2 \ldots, k$; $m, n = 0, \pm 1, \pm \ldots$). We denote the union of all the L^j_{mn} within the limits of a group by l_{mn}, and the unbounded domain occupied by the homogeneous medium by D. Then the complete boundary of D is $L = \mathrm{U}\, l_{m,n}$. We place the origin inside the domain $D_{00}{}^0$, contained in the fundamental period parallelogram Π_{00}. Because of the congruence of the groups we have $L^j_{mn} \equiv L_{00}{}^j \,(\mathrm{mod}\,\omega_1, \omega_2)$. For a fixed j, each system of congruent inclusions D^j_{mn} is characterized by any of their representative, for example, $D_{00}{}^j$, with modulus of elasticity E_j and Poisson ratio μ_j ($j = 1, 2 \ldots, k$). We denote the modulus of elasticity and the Poisson's ratio for the fundamental medium by E and μ, respectively.

We assume that within the limits of the period parallelogram Π_{mn} the average stresses which act are S_1, S_2 and S_{12}, that the stress vector varies continuously at the passage from D to D^j_{mn} ($j = 1,2, \ldots, k$; $m, n = 0, \pm 1, \pm, \ldots$), and that the displacement vector undergoes a discontinuity $g_j(t)$, $t \in L^j_{mn}$ (Fig. 1).

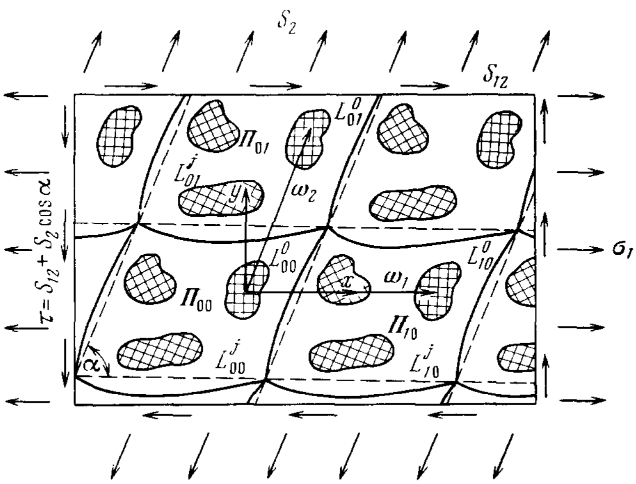

Fig. 1

The problem consists in the description of the state of stress of the structure under consideration and also in the construction of a homogeneous anisotropic model medium. The latter forms the substance of the reduction problem for the structure and consists essentially in the establishment of the relation between the average stresses and the average strains.

The assignment of the same average stresses to all parallelograms Π_{mn} implies the periodic character of the stresses and, under additional conditions, the periodicity of the rotations and the quasi-periodicity of the displacements in D.

Indeed, the resultant force, acting along the arbitrary curve AB at D, has the form [4]

$$X + iY = \int\limits_{AB} (X_n + iY_n)\, ds = -ig(z)\big|_A^B, \quad g(z) = \varphi(z) + z\overline{\varphi'(z)} + \overline{\psi(z)} \tag{1.1}$$

Then the static conditions, ensuring identical average stresses for each period prallelogram, can be represented in the form

$$g(z + \omega_1) - g(z) = -i(S_{12} + S_2 e^{i\alpha})\omega_1, \quad \alpha = \arg\omega_2 \tag{1.2}$$

$$g(z + \omega_2) - g(z) = i(S_1 + S_{12} e^{i\alpha})\,|\,\omega_2\,|$$

From the identities (1.2) we obtain the quasi-periodicity of the function $g(z)$, while their differentiation with respect to z and $\bar{z}$ leads us to the relations

$$\operatorname{Re}\varphi'(z)\big|_z^{z+\omega_\nu} = 0, \quad (\bar{z}\varphi''(z) + \Psi'(z))\big|_z^{z+\omega_\nu} = 0, \quad \nu = 1, 2 \tag{1.3}$$

From here we obtain the periodicity of the stresses in D. The quasi-periodicity of the displacements follows from the formulas [4]

$$h(z) = 2G(u + iv) = \varkappa\varphi(z) - z\overline{\varphi'(z)} - \overline{\psi(z)} = (\varkappa + 1)\varphi(z) - g(z) \tag{1.4}$$

taking into account that by virtue of (1.3) and of the assumption on the periodicity of the rotations, the function $\varphi(z)$ is quasi-periodic. Conversely, one can show that from the condition of the quasi-periodicity of the displacements there follows the periodicity of the stresses and the quasi-periodicity of the function $g(z)$ in D.

Proceeding from the stated considerations, the formulation of the fundamental problem can be made in the following manner: construct functions $\varphi(z)$, $\psi(z)$ and $\varphi_j(z)$, $\psi_j(z)$, regular in the domains D, $D_{00}{}^j$ ($j = 1, 2, \ldots, k$) and satisfying on l_{00} the conditions of the junction of the media and of the inclusions

$$\varphi(t) + t\overline{\varphi'(t)} + \overline{\psi(t)} = \varphi_j(t) + t\overline{\varphi_j'(t)} + \overline{\psi_j(t)}, \quad t \in L_{00}{}^j \tag{1.5}$$

$$\frac{\varkappa}{G}\varphi(t) - \frac{1}{G}\{t\overline{\varphi'(t)} + \overline{\psi(t)}\} = \frac{\varkappa_j}{G_j}\varphi_j(t) - \frac{1}{G_j}\{t\overline{\varphi_j'(t)} + \overline{\psi_j(t)}\} + 2g_j(t)$$

$$G = \frac{E}{2(1+\mu)}, \quad G_j = \frac{E_j}{2(1+\mu_j)}, \quad \varkappa = \frac{3-\mu}{1+\mu}, \quad \varkappa_j = \frac{3-\mu_j}{1+\mu_j}, \quad j = 1, 2, \ldots, k$$

(in the case of plane deformation $\varkappa = 3-4\mu$, $\varkappa_j = 3-4\mu_j$) and the static conditions (1.2). Obviously, it is understood that all the periodicity conditions are satisfied automatically at the expense of the special form of representation of the desired regular functions.

We carry out these representations by making use of ideas contained in [5] and of the solving scheme of the first fundamental doubly-periodic problem of the theory of elasticity, developed in [3]. We write

$$\varphi(z) = \frac{1}{2\pi i}\int\limits_{l_{00}} p(t)\zeta(t-z)\,dt + Az, \quad z \in D_{00}{}^j, \quad j = 1, 2, \ldots, k \tag{1.6}$$

$$\psi(z) = \frac{1}{2\pi i}\int\limits_{l_{00}} \{\varepsilon(t)\overline{p(t)} - \bar{t}p'(t) + r(t)\overline{q(t)}\}\zeta(t-z)\,dt +$$

$$\frac{1}{2\pi i}\int\limits_{l_{00}} p(t)\rho_1(t-z)\,dt + Bz$$

$$\varphi_j(z)=\frac{1}{2\pi i}\int\limits_{L_{00}^j}\frac{q_j(t)}{t-z}\,dt,\quad \psi_j(z)=\frac{1}{2\pi i}\int\limits_{L_{00}^j}\frac{\alpha_j\overline{p_j(t)}+\beta_j\overline{q_j(t)}-\bar{t}q_j'(t)}{t-z}\,dt$$

Here $\zeta(z)$ is the Weierstrass zeta-function [6], ρ_1 is a special meromorphic function [3, 7], $p(t)=\{p_j(t),\ t\in L_{00}^j\}$ and $q(t)=\{q_j(t),\ t\in L_{00}^j\}$ are, in general, complex functions subject to determination on l_{00}. The piecewise constants $\varepsilon(t)=\{\varepsilon_j,\ t\in L_{00}^j\}$, $r(t)=\{r_j,\ t\in L_{00}^j\}$ and the constants A, B, α_j and β_j ($j=1,2,\ldots,k$) are arbitrary for the present. The integration is taken in the counterclockwise direction. The representations (1.6) guarantee the double periodicity of the stresses and the quasi-periodicity of the displacements in D. This follows immediately from the quasi-periodicity of the Weierstrass zeta-function and from the realtions [7]

$$\rho_1(z+\omega_\nu)-\rho_1(z)=\bar{\omega}_\nu\rho(z)+\gamma_\nu$$

$$\gamma_\nu=2\rho_1\left(\frac{\omega_\nu}{2}\right)-\bar{\omega}_\nu\rho\left(\frac{\omega_\nu}{2}\right),\quad \nu=1,2 \tag{1.7}$$

where $\rho(z)$ is the Weierstrass elliptic function.

We determine the constants A and B occurring in (1.6) so that the static conditions (1.2) should be satisfied. Making use of (1.6) and of the formulas (1.7), we reduce (1.2) to a system of two equations in A and B

$$\begin{gathered}(A+\bar{A})\omega_1+\bar{B}\bar{\omega}_1+\delta_1 b+\bar{\gamma}_1\bar{b}-\bar{\delta}_1\bar{a}=-i\omega_1(S_{12}+S_2e^{i\alpha})\\ (A+\bar{A})\omega_2+\bar{B}\bar{\omega}_2+\delta_2 b+\bar{\gamma}_2\bar{b}-\bar{\delta}_2\bar{a}=i|\omega_2|(S_1+S_{12}e^{i\alpha})\\ a=\frac{1}{2\pi i}\int\limits_{l_{00}}\{(\varepsilon(t)\overline{p(t)}+r(t)\overline{q(t)}\}\,dt+\frac{1}{2\pi i}\int\limits_{l_{00}}p(t)\,\overline{dt}\\ b=-\frac{1}{2\pi i}\int\limits_{l_{00}}p(t)\,dt,\quad \delta_1=2\zeta\left(\frac{\omega_1}{2}\right),\quad \delta_2=2\zeta\left(\frac{\omega_2}{2}\right)\end{gathered} \tag{1.8}$$

The solution of this system has the form

$$\begin{gathered}B=B_L-\frac{1}{2\sin\alpha}(S_1+2S_{12}e^{-i\alpha}+S_2e^{-2i\alpha}),\quad \operatorname{Re}A_L=\operatorname{Re}\left(\frac{\pi}{2S}a+\frac{\pi}{S}b-\frac{\delta_1 b}{\omega_1}\right)\\ \operatorname{Re}A=\operatorname{Re}A_L+\frac{1}{4\sin\alpha}(S_1+2S_{12}\cos\alpha+S_2),\quad S=\omega_1\operatorname{Im}\omega_2\\ B_L=\frac{\delta_1-\gamma_1}{\omega_1}b-\frac{2\pi}{S}\operatorname{Re}b-\left(\frac{\pi}{S}-\frac{\delta_1}{\omega_1}\right)\operatorname{Re}a,\quad \alpha=\arg\omega_2\end{gathered} \tag{1.9}$$

The compatibility condition of the system (1.8) is the equality

$$2\pi\operatorname{Im}a=\operatorname{Re}\int\limits_{l_{00}}\{(\varepsilon(t)\overline{p(t)}+r(t)\overline{q(t)}+p(t)\overline{dt}\}=0 \tag{1.10}$$

Let us give its mechanical interpretation. To this end we consider the expression of the principal moment of the forces acting along l_{00} from the side of the domain D. We have [4]

$$M=\operatorname{Re}\int\limits_{l_{00}}q(t)\,\overline{dt}=\operatorname{Re}\int\limits_{l_{00}}\{\varphi(t)+t\,\overline{\varphi'(t)}+\overline{\psi(t)}\}\,dt \tag{1.11}$$

$$j=1,2,\ldots,k$$

Taking the limit in the second of the formulas (1.6), we find

$$\psi(t) = \psi^+(t) - (\varepsilon(t)\,\overline{p(t)} - \bar{t}p'(t) + r(t)\,\overline{q(t)}),\quad t \in L_{00}{}^j,\quad j = 1, 2, \ldots, k \quad (1.12)$$

Here $\psi(t)$ is the limiting value of the function, regular in D, while $\psi^+(t)$ are the limiting values of the functions, regular in the simply connected domains $\bar{D}_{00}{}^j$. Integrating by parts in (1.11) and substituting for $\psi(t)$ its value from (1.12), we obtain

$$M = -\operatorname{Re}\int_{l_{00}} \{(\varepsilon(t)\,\overline{p(t)} + r(t)\,\overline{q(t)})\,dt + p(t)\,\overline{dt}\} \quad (1.13)$$

From the formula (1.13) it follows that the compatibility condition (1.10) guarantees the vanishing of the principal moment of the forces acting on the boundary l_{00} from the side of the domain D. From here, in particular, it follows that under the condition (1.10), the principal moment of all the forces acting along the boundary of the period parallelogram or of any other fundamental domain, containing l_{00} is also equal to zero.

Thus, the representations (1.6) together with the formulas (1.9) and the additional condition (1.10) guarantee the doubly-periodic distribution of the stresses and the existence of the given average stresses in the structure. The problem reduces therefore to the determination on $L_{00}{}^j$ of the densities $p_j(t)$, $q_j(t)$ ($j = 1, 2, \ldots, k$) from the boundary conditions (1.5). Simultaneously, it is necessary to find the constants ε_j, r_j, α_j, β_j and $\operatorname{Im} A$, which have remained undetermined after satisfying the static conditions. This latter constant is given by the following functional:

$$\operatorname{Im} A = \operatorname{Im}\left\{\left(\frac{\pi}{S} - \frac{\delta_1}{\omega_1}\right) b\right\} \quad (1.14)$$

Here S is the area of the period parallelogram, b is the functional given in (1.8). The mechanical interpretation of the formula (1.14) will be given in Sect. 5.

2. The solution of the boundary value problem (1.5). We reduce the boundary value problem (1.5) to the equivalent system of Fredholm integral equations of the second kind. To this end we pass in the representations (1.6) to the corresponding limiting values and we insert them into the boundary conditions (1.5). The system of integral equations relative to $p_j(t)$, $q_j(t)$, obtained in this way, will be a Fredholm system if we set

$$\alpha_j = \frac{1+\varkappa}{\lambda_j - 1},\quad \beta_j = \frac{1+\varkappa_j\lambda_j}{1-\lambda_j},\quad \varepsilon_j = \frac{\varkappa+\lambda_j}{\lambda_j - 1},\quad r_j = \frac{(1+\varkappa_j)\lambda_j}{1-\lambda_j},\quad \lambda_j = \frac{G}{G_j} \quad (2.1)$$

It has the form

$$p(t_0) - M_j\{p(t), q(t), t_0\} = P_j(t_0),\quad t_0 \in L_{00}{}^j$$
$$q(t_0) - N_j\{q(t), p(t), t_0\} = Q_j(t_0),\quad j = 1, 2, \ldots, k \quad (2.2)$$

Here

$$M_j\{p(t), q(t), t_0\} = \frac{1}{2\pi i}\int_{L_{00}{}^j}\left\{p(t)\,d\left[\ln\frac{\sigma(t-t_0)}{\overline{\sigma(t-t_0)}}\right] + \frac{r_j}{\varepsilon_j}q(t)\,d\left[\ln\frac{t-t_0}{\overline{\sigma(t-t_0)}}\right]\right\} -$$
$$\frac{1}{2\pi i\varepsilon_j}\int_{l_{00}}\overline{p(t)}\,d\{(t-t_0)\,\overline{\zeta(t-t_0)} - \zeta_1\overline{(t-t_0)}\} -$$
$$\frac{1}{2\pi i}\int_{l_{00}^{*j}}\left\{p(t)\left(\frac{\varepsilon}{\varepsilon_j}\overline{\zeta(t-t_0)}\,\overline{dt} - \zeta(t-t_0)\,dt\right) + \frac{r}{\varepsilon_j}q(t)\,\overline{\zeta(t-t_0)}\,\overline{dt}\right\} +$$
$$t_0\left\{\left(1+\frac{1}{\varepsilon_j}\right)\operatorname{Re} A_L + i\left(1-\frac{1}{\varepsilon_j}\right)\operatorname{Im} A\right\} + \frac{1}{\varepsilon_j}\bar{t}_0 B_L$$

$$N_j\{q)t),\ p(t),\ t_0\} = \frac{1}{2\pi i}\int\limits_{L_{00}^j}\left\{q(t)\,d\left[\ln\frac{\bar t-\bar t_0}{t-t_0}\right] + \frac{1}{\beta_j}\overline{q(t)}\,d\left[\frac{t-t_0}{\bar t-\bar t_0}\right] +\right.$$

$$\left.\frac{\alpha_j}{\beta_j}p(t)\,d\left[\ln\frac{\bar t-\bar t_0}{\sigma(t-t_0)}\right]\right\} - \frac{1}{2\pi i}\int\limits_{l_{00}^{*j}}\frac{\alpha_j}{\beta_j}p(t)\zeta(t-t_0)\,dt - \frac{\alpha_j}{\beta_j}A_L t_0$$

$$P_j(t_0) = \left(1+\frac{1}{\varepsilon_j}\right)\frac{\sigma_1+\sigma_2}{4}t_0 + \frac{\sigma_2-\sigma_1-2i\tau}{2\varepsilon_j}\bar t_0 + \frac{2G}{\varkappa+\lambda_j}g_j(t_0)$$

$$Q_j(t_0) = -\frac{\sigma_1+\sigma_2}{4}\frac{\alpha_j}{\beta_j}t_0 - \frac{2G}{1+\varkappa_j\lambda_j}g_j(t_0)$$

$$\sigma_1\sin\alpha = S_1 + 2S_{12}\cos\alpha + S_2\cos^2\alpha,\quad \tau = S_{12}+S_2\cos\alpha$$

$$\sigma_2 = S_2\sin\alpha,\quad l_{00} = \bigcup_{j=1}^{k} L_{00}{}^j,\quad l_{00}{}^{*j} = l_{00}\setminus L_{00}{}^j$$

The functionals $\mathrm{Re}A_L$, B_L are defined in (1.9), the functional $\mathrm{Im}A$ in (1.14), the constants λ_j, α_j, β_j, ε_j, r_j are expressed in terms of the elastic characteristics of the components of the structure by the formulas (2.1). The quantities σ_1, σ_2 and τ are the average normal and tangential stresses on areas perpendicular to the coordinate axes ox and oy.

If $k=1$, i.e. within the limits of the period parallelogram there exists only one inclusion, then the terms in M_j and N_j, which contain integrals along the agregate of the contours l_{00}^{*j}, will vanish. In this case $l_{00} = L_{00}{}^j$ $(j=1)$.

It is important to note that every solution of the obtained system of equations (2.2) satisfy the additional condition (1.10). Indeed, we multiply the first of the equalities in (1.5) by $\overline{dt}$ and we integrate along the contour $L_{00}{}^j$ $(j = 1,2\ldots, k)$. We obtain, by virtue of the regularity of $\psi_j(z)$ in $D_{00}{}^j$

$$\mathrm{Re}\int\limits_{l_{00}}\{\varphi(t)+t\overline{\varphi'(t)}+\overline{\psi(t)}\}d\bar t = 0 \tag{2.3}$$

From this, taking into account the equalities (1.13) and (1.11), we arrive at the condition (1.10).

3. The uniqueness theorem. By a fundamental cell of the structure we will understand any fundamental domain D_{00} in it, which contains all the continua $D_{00}{}^j$ $(j = 1,2,\ldots,k)$. In particular, this may be the basic period parallelogram Π_{00} with the boundary Γ.

We consider the potential energy of deformation of the fundamental cell. We have

$$2\iint\limits_{D_\Gamma} W\,dx\,dy + 2\sum_{j=1}^{k}\iint\limits_{D_{00}{}^j} W_j\,dx\,dy =$$

$$\int\limits_{l_{00}+\Gamma}(X_n u + Y_n v)\,ds - \sum_{j=1}^{k}\int\limits_{L_{00}{}^j}(X_n{}^j u_j + Y_n{}^j v_j)\,ds$$

$$D_\Gamma = D_{00}\setminus\bigcup_{j=1}^{k}D_{00}{}^j \tag{3.1}$$

Here W, W_j are known positive definite quadratic forms in the components of the strain or of the stress, X_n, Y_n are the components of the stress vector acting on $l_{00}+\Gamma$

from the side of the domain D; $X_n{}^j$, $Y_n{}^j$ are the components of the stress vector acting on $L_{00}{}^j$ from the side of $D_{00}{}^j$; u, v and u_j, v_j are the displacements along $L_{00}{}^j$, correspondingly from the sides of D and $D_{00}{}^j$. The integration in the right-hand side of (3.1) is taken in such a way that the domain D_Γ is at the left-hand side when moving along its boundary. Taking into account the quasi-periodicity of the displacements, the formula (1.2) and the fact that the stress vector is continued across l_{00} uninterruptedly, while the displacement vector undergoes a jump $g_* = \{g_j(t),\ t \in L_{00}{}^j\}$, we rewrite (3.1) in the form

$$2\int\int_{D_\Gamma} W\,dx\,dy + 2\sum_{j=1}^{k}\int\int_{D_{00}{}^j} W_j\,dx\,dy = \operatorname{Re}\int_{l_{00}} (X_n - iY_n)\,g_*(t)\,ds + \qquad (3.2)$$

$$\operatorname{Re}\{(S_{12} + S_2 e^{-i\alpha})\,\omega_1\Omega_2 + (S_1 + S_{12}e^{-i\alpha})\,|\,\omega_2\,|\,\Omega_1\}$$

$$\Omega_\nu = (u + iv)\,\Big|_z^{z+\omega_\nu}, \qquad \nu = 1, 2$$

We apply formula (3.2) to the difference of two solutions, each of which satisfies the boundary conditions (1.5) and the static conditions (1.2). Obviously, this solution corresponds to the boundary value problem with

$$g_*(t) = 0, \quad S_1 = S_2 = S_{12} = 0 \qquad (3.3)$$

From (3.3) there follows the vanishing of the right-hand side of (3.2), which shows the uniqueness theorem for solutions in D and in the domains D^j_{mn} $(m, n = 0, \pm 1, \pm, \ldots;\ j = 1, 2, \ldots, k)$. From the uniqueness theorem it follows, in particular, that the solution of the boundary value problem (1.5) under the conditions (3.3) can be represented in the form

$$\varphi^0(z) = iCz + E, \qquad \psi^0(z) = -\bar{E}, \qquad \operatorname{Im} C = 0$$

$$\varphi_j{}^0(z) = iC_j z + E_j, \quad \psi_j{}^0(z) = -\bar{E}_j, \quad \operatorname{Im} C_j = 0, \quad j = 1, 2, \ldots, k \qquad (3.4)$$

$$\frac{\varkappa + 1}{G}\,C = \frac{\varkappa_j + 1}{G_j}\,C_j, \qquad \frac{\varkappa + 1}{G}\,E = \frac{\varkappa_j + 1}{G_j}\,E_j$$

Here E, E_j are, in general, complex constants. The formulas (3.4) coincide with the solutions of the corresponding homogeneous problem for a finite multi-connected domain [5].

4. The existence of the solution. We will assume that the function $g_*(t)$ is differentiable and its derivative satisfies the Hölder condition. This is sufficient in order that the solutions $p(t)$ and $q(t)$ be differentiable and that their derivatives also satisfy the Hölder condition on l_{00} [4].

We prove that under these conditions the system of integral equations (2.2) is always solvable.

Obviously, for $P_j(t) = 0$, $Q_j(t) = 0$ it is necessary and sufficient that

$$S_1 = S_2 = S_{12} = 0, \qquad g_j(t) = 0 \qquad (4.1)$$

Thus, the integral equations (2.2) with zero right-hand sides correspond to the boundary value problem (1.5) with zero average stresses and $g_j(t) = 0$, $j = 1, 2, \ldots, k$.

We denote the solution of these homogeneous integral equations by $p_0(t) =$

$\{p_j^0(t),\ t \in L_{00}^j\}$ and $q_0(t) = \{q_j^0(t), t \in L_{00}^j\}$. According to (1.6), their corresponding regular functions can be written in the form

$$\varphi_0(z) = \frac{1}{2\pi i}\int_{l_{00}} p_0(t)\zeta(t-z)\,dt + A_0 z, \quad z \in D_{00}^j, \quad j = 1, 2, \ldots, k$$

$$\psi_0(z) = \frac{1}{2\pi i}\int_{l_{00}} \{\varepsilon(t)\overline{p_0(t)} - \bar{t}p_0'(t) + r(t)\overline{q_0(t)}\}\zeta(t-z)\,dt +$$

$$\frac{1}{2\pi i}\int_{l_{00}} p_0(t)\rho_1(t-z)\,dt + B_0 z \tag{4.2}$$

$$\varphi_j^0(z) = \frac{1}{2\pi i}\int_{L_{00}^j} \frac{q_j^0(t)}{t-z}\,dt, \ \psi_j^0(z) = \frac{1}{2\pi i}\int_{L_{00}^j}\left[\alpha_j\overline{p_j^0(t)} + \beta_j\overline{q_j^0(t)} - \bar{t}\frac{d}{dt}q_j^0(t)\right]\frac{dt}{t-z}$$

Here A_0, B_0 are determined by the formulas (1.9) and (1.14), in which everywhere instead of $p(t)$ and $q(t)$ we have $P_0(t)$ and $q_0(t)$, respectively. All the remaining functionals which occur below and which correspond to the solutions of the homogeneous equations, will also carry a zero subscript.

Comparing the identically named functions from (3.4) and (4.2), we arrive to the relations

$$\frac{1}{2\pi i}\int_{l_{00}} p_0(t)\zeta(t-z)\,dt + A_0 z - iCz - E = 0, \quad z \in D$$

$$\frac{1}{2\pi i}\int_{l_{00}} p_0(t)\rho_1(t-z)\,dt +$$

$$\frac{1}{2\pi i}\int_{l_{00}} \{\varepsilon(t)\overline{p_0(t)} - \bar{t}p_0'(t) + r(t)\overline{q_0(t)}\}\zeta(t-z)\,dt + B_0 z + \bar{E} = 0$$

$$\frac{1}{2\pi i}\int_{L_{00}^j} \frac{q_{j0}(t)}{t-z}\,dt - iC_j z - E_j = 0, \quad z \in D_{00}^j, \quad j = 1, 2, \ldots, k \tag{4.3}$$

$$\frac{1}{2\pi i}\int_{L_{00}^j}\left[\alpha_j\overline{p_j^0(t)} + \beta_j\overline{q_j^0(t)} - \bar{t}\frac{d}{dt}q_j^0(t)\right]\frac{dt}{t-z} + \bar{E}_j = 0$$

Between the constants C and C_j, E and E_j we have the relations given in (3.4). Computing the increase of the left-hand side of the first equality in (4.3) when we pass from point z to $z + \omega_\nu$ $(\nu = 1,2)$, we obtain, taking into account (1.8), (1.14) and (3.4),

$$\frac{1}{2\pi i}\int_{l_{00}} p_0(t)\zeta(t-z)\,dt = E, \quad b_0 = 0, \quad A_0 = iC = 0, \quad C_j = 0, \tag{4.4}$$

$$j = 1, 2, \ldots; k; \quad z \in D$$

Since $\mathrm{Re}A_0 = 0$, $b_0 = 0$, from (1.9) we have

$$a_0 = \frac{1}{2\pi i}\int_{l_{00}}(\varepsilon(t)\overline{p_0(t)} + r(t)\overline{q_0(t)})\,dt + \frac{1}{2\pi i}\int_{l_{00}} p_0(t)\,\overline{dt} = 0,\ B_0 = 0 \tag{4.5}$$

We consider the piecewise analytic function

$$\Phi(z) = \frac{1}{2\pi i}\int\limits_{l_{00}} p_0(t)\,\zeta(t-z)\,dt - E \tag{4.6}$$

Computing the jump of the limiting values of the function $\Phi(z)$ on l_{00} and taking into account the first equality in (4.4), we arrive at the conclusion that $p_0(t)$ will be the boundary value of some functions, regular in the domains $D_{00}{}^j$. In such case the integral in (4.4) vanishes and we can write in view of (4.4) and (3.4)

$$E = 0,\ E_j = 0, \qquad j = 1, 2, \ldots, k \tag{4.7}$$

Because of the regularity of the function $\rho_1(z)$ in the basic period parallelogram [2] and because of the above established property of $p_0(t)$, we have the equality

$$\int\limits_{l_{00}} p_0(t)\,\rho_1(t-z)\,dt = 0 \tag{4.8}$$

We introduce now the following functions:

$$i\chi_j(t) = p_j{}^0(t), \quad i\delta_j(t) = \varepsilon_j\overline{p_j{}^0(t)} + r_j\overline{q_j{}^0(t)} - \bar{t}\frac{d}{dt}p_j{}^0(t), \quad t \in L_{00}{}^j \tag{4.9}$$

$$i\theta_j(t) = q_j{}^0(t), \quad i\sigma_j(t) = \alpha_j\overline{p_j{}^0(t)} + \beta_j\overline{q_j{}^0(t)} - \bar{t}\frac{d}{dt}q_j{}^0(t), \quad j = 1, 2, \ldots, k$$

From the first two equalities in (4.3), taking into account (4.4), (4.5), (4.7) and (4.8), it follows that the functions $\chi_j(t)$ and $\delta_j(t)$ are the boundary values of the functions $\chi_j(z)$ and $\delta_j(z)$, regular in the domains $D_{00}{}^j$ $(j = 1, 2, \ldots, k)$. The remaining two equalities in (4.3) show that $\theta_j(t)$ and $\sigma_j(t)$ are the boundary values of the functions $\theta_j(z)$ and $\sigma_j(z)$, regular in the domain $D \setminus D_{00}{}^j$ $(j = 1, 2, \ldots, k)$ and vanishing at infinity. Eliminating the functions $p_0(t)$ and $q_0(t)$ from the relations (4.9), we obtain the system of equations

$$\overline{\chi_j(t)} + \bar{t}\chi_j'(t) + \delta_j(t) = \overline{\theta_j(t)} + \bar{t}\theta_j'(t) + \sigma_j(t), \quad t \in L_{00}{}^j, \quad j = 1, 2, \ldots, k$$

$$\frac{\varkappa}{G}\overline{\chi_j(t)} - \frac{1}{G}\{\bar{t}\chi_j'(t) + \delta_j(t)\} = \frac{\varkappa_j}{G_j}\overline{\theta_j(t)} - \frac{1}{G_j}\{\bar{t}\theta_j'(t) + \sigma_j(t)\} \tag{4.10}$$

From Eqs. (4.10) it follows that for each fixed j, the functions $\chi_j(z)$, $\delta_j(z)$, $\theta_j(z)$ and $\sigma_j(z)$ give the solution of the problem on the elastic equilibrium of an unbounded nonhomogeneous medium with the separation line $L_{00}{}^j$. In this case, G_j and $\varkappa_j$ are the elastic characteristics of the medium which occupies the domain $D_{00}{}^j$; G and $\varkappa$ are the corresponding characteristics in the domain $D \setminus D_{00}{}^j$; the stress and the elastic displacement vectors vary continuously at the passage through $L_{00}{}^j$ and at infinity both stresses and displacements are equal to zero.

Such a boundary value problem has only the trivial solution [5]

$$\theta_j(z) = 0,\ \sigma_j(z) = 0,\ \chi_j(z) = 0,\ \delta_j(z) = 0, \qquad j = 1, 2, \ldots, k \tag{4.11}$$

From (4.9) and (4.11) we obtain

$$p_j{}^0(t) = q_j{}^0(t) = 0, \qquad j = 1, 2, \ldots, k \tag{4.12}$$

Thus, we have proved the existence and the uniqueness of the solution of the system of Fredholm integral equations (2.2).

In the limiting case $\omega_1 \to \infty$ and $\omega_2 \to \infty$, we obtain the unbounded medium with the group of inclusions $D_{00}{}^j$ $(j = 1,2, \ldots, k)$. The formulas,obtained above for a regular structure, hold also (after the corresponding limiting process) for this degenrate case.

5. A macroscopic model of the structure. By the average strains in a structure we will understand the strains in its fundamental cell. Since any two congruent cells have the same strains, we can introduce a model in the following way.

Definition. By the model of a regular structure we mean an elastic homogeneous medium having the property that whenever the tensors of the average stresses, which act in the structure and in the model, coincide, the corresponding tensors of the average strains also coincide.

Setting $q_*(t) = 0$, we find the relation between the average stresses and the average strains in the structure.

From the formulas (1.4), taking into account (1.2) and (1.6), we find the increments of the displacements u and v

$$\begin{gathered}
2G\,[u\,(z + \omega_1) - u\,(z)] = (\varkappa + 1)\ \mathrm{Re}\,(b\delta_1 + A\,\omega_1) - S_2\omega_1 \sin\alpha \\
2G\,[v\,(z + \omega_1) - v\,(z)] = (\varkappa + 1)\ \mathrm{Im}\,(b\delta_1 + A\,\omega_1) + \omega_1\,(S_{12} + S_2 \cos\ \alpha) \\
2G\,[u(z + \omega_2) - u\,(z)] = (\varkappa + 1)\ [\mathrm{Re}\,(b\delta_2 + Ah) - H\,\mathrm{Im}A] + |\omega_2| S_{12} \sin\alpha \\
2G\,[v\,(z + \omega_2) - v\,(z)] = (\varkappa + 1)\ \ [\mathrm{Im}\,(b\delta_2 + Ah) + H\,\mathrm{Re}A] - |\omega_2|\,(S_1 + S_{12}\cos\alpha) \\
h = \mathrm{Re}\,\omega_2, \quad H = \mathrm{Im}\,\omega_2
\end{gathered} \tag{5.1}$$

On the other hand, the displacements of the point z relative to its congruent point $z + \omega_\nu$ $(\nu = 1, 2)$ are connected with the average strains e_1, e_2, e_{12} and the rotation ω of the fundamental cell in the following manner:

$$u\,(z + \omega_1) - u\,(z) = \omega_1 e_1, \quad v\,(z + \omega_1) - v\,(z) = \omega_1\,(e_{12} + \omega) \tag{5.2}$$
$$u\,(z + \omega_2) - u\,(z) = he_1 + H\,(e_{12} - \omega), \quad v\,(z + \omega_2) - v\,(z) = h\,(e_{12} + \omega) + He_2$$

Inserting (5.2) into the left-hand sides of (5.1) and solving the obtained system of equations relative to e_1, e_2, e_{12} and ω, we find

$$\begin{gathered}
2Ge_1 = (\varkappa + 1)\,\mathrm{Re}\left(\frac{b\delta_1}{\omega_1} + A\right) - S_2 \sin\alpha \\
2Ge_2 = (\varkappa + 1)\left\{\mathrm{Im}\left(\frac{b\delta_2}{H} - \frac{b\delta_1}{\omega_1}\,\mathrm{ctg}\,\alpha\right) + \mathrm{Re}\,A\right\} - \frac{S_1 + 2S_{12}\cos\alpha + S_2\cos^2\alpha}{\sin\alpha} \\
2Ge_{12} = (\varkappa + 1)\ \left\{\mathrm{Re}\left(\frac{b\delta_2}{2H} - \frac{b\delta_1}{2\omega_1}\,\mathrm{ctg}\,\alpha\right) + \mathrm{Im}\,\frac{b\delta_1}{2\omega_1}\right\} + S_{12} + S_2\cos\alpha
\end{gathered} \tag{5.3}$$

$$2G\omega = (\varkappa + 1)\left\{\mathrm{Re}\left(\frac{b\delta_1}{2\omega_1}\,\mathrm{ctg}\,\alpha - \frac{b\delta_2}{2H}\right) + \mathrm{Im}\,\frac{b\delta_1}{2\omega_1} + \mathrm{Im}\,A\right\} \tag{5.4}$$

We require that the rotation of the fundamental cell be equal to zero. This condition can be satisfied at the expense of $\mathrm{Im}A$, which occurs in (5.4). Determining $\mathrm{Im}A$ from (5.4) and taking into account Legendre's relation $\delta_1\omega_2 - \delta_2\omega_1 = 2\pi i$ [6], we obtain formula (1.14). From here, its mechanical interpretation is obvious. We introduce the standard solutions of the system of equations (2.2) p_{ik}, q_{ik} $(i, k = 1,2)$, defined by the formulas

$$p(t) = \sigma_1 p_{11}(t) + \tau p_{12}(t) + \sigma_2 p_{22}(t) \tag{5.5}$$

$$q(t) = \sigma_1 q_{11}(t) + \tau q_{12}(t) + \sigma_2 q_{22}(t)$$

Due to (5. 5), the functionals a and b, defined in (1. 8), can be represented in the form

$$a = \sigma_1 a_{11} + \tau a_{12} + \sigma_2 a_{22}, \quad b = \sigma_1 b_{11} + \tau b_{12} + \sigma_2 b_{22} \tag{5.6}$$

Here σ_1, σ_2 and τ are the average stresses on the areas perpendicular to the coordinate axes. and a_{ik}, b_{ik} are the functionals corresponding to the standard solutions $p_{ik}(t)$, $q_{ik}(t)$.

Inserting into (5. 9) for $\mathrm{Re}A$, its value from (1. 9), making use of the equalities (5.6) and the relations between S_1, S_2, S_{12} and σ_1, σ_2, τ, given in (2. 2), we obtain the relations between the average stresses and the average strains in the structure

$$\begin{aligned} e_1 &= \sigma_1 \left\{\frac{1}{E} + \frac{2\pi}{ES} \mathrm{Re}(a_{11} + 2b_{11})\right\} + \sigma_2 \left\{\frac{2\pi}{ES} \mathrm{Re}(a_{22} + 2b_{22}) - \frac{\mu}{E}\right\} + \\ &\qquad \tau\left\{\frac{2\pi}{ES} \mathrm{Re}(a_{12} + 2b_{12})\right\} \\ e_2 &= \sigma_1 \left\{\frac{2\pi}{ES} \mathrm{Re}(a_{11} - 2b_{11}) - \frac{\mu}{E}\right\} + \sigma_2 \left\{\frac{1}{E} + \frac{2\pi}{ES} \mathrm{Re}(a_{22} - 2b_{22})\right\} + \\ &\qquad \tau\left\{\frac{2\pi}{ES} \mathrm{Re}(a_{12} - 2b_{12})\right\} \\ 2e_{12} &= \sigma_1 \left\{\frac{8\pi}{ES} \mathrm{Im}\, b_{11}\right\} + \sigma_2 \left\{\frac{8\pi}{ES} \mathrm{Im}\, b_{22}\right\} + \tau\left\{2\,\frac{1+\mu}{E} + \frac{8\pi}{ES} \mathrm{Im}\, b_{12}\right\} \end{aligned} \tag{5.7}$$

The expressions in the braces represent the macroscopic elastic parameters of the structure.

We return to the energy relation (3. 2). Taking into accont the relations (5. 2) and the fact that in our case $q_*(t) = 0$, we write it in the form

$$\Pi = \iint\limits_D W dxdy + \sum_{j=1}^{k} \iint\limits_{D_{00}^j} W_j dxdy = \frac{S}{2}(e_1\sigma_1 + 2e_{12}\tau + e_2\sigma_2) \tag{5.8}$$

From here it follows that a regular elastic structure and its model are energetically identical. The relation (5. 8) could have been taken as the definition of the model medium. The matrix of the macroscopic elastic parameters in (5. 7) is symmetric and energetically admissible.

Indeed, assume that the i th state of the system with the components $\sigma_x{}^i$, $\sigma_y{}^i$, $\tau_{xy}{}^i$, $e_x{}^i$, $e_y{}^i$, $e_{xy}{}^i$ corresponds to the situation when only one average stress $\sigma_i = 1$ acts (from three possible: σ_1, σ_2 and $\tau = \sigma_3$). We denote by Π_{ik} that part of the potential energy Π, which corresponds to the work done by the stresses of the i th state and the strains of the k th state. Then formula (5. 8) can be represented in the form

$$\sum_{i,k=1}^{3} \sigma_i \sigma_k \Pi_{ik} = S/2\,(\sigma_1 e_1 + 2\sigma_3 e_{12} + \sigma_2 e_2), \qquad \Pi_{ik} = \Pi_{ki} \tag{5.9}$$

Differentiating (5. 9) successively with respect to σ_i $(i = 1, 2, 3)$, we find

$$\begin{gathered} e_1 = 2/S\,(\sigma_1\Pi_{11} + \sigma_2\Pi_{12} + \tau\Pi_{13}), \qquad e_2 = 2/S\,(\sigma_1\Pi_{21} + \sigma_2\Pi_{22} + \tau\Pi_{23}) \\ 2e_{12} = 2/S\,(\sigma_1\Pi_{31} + \sigma_2\Pi_{32} + \tau\Pi_{33}), \qquad \Pi_{ii} > 0 \end{gathered} \tag{5.10}$$

Comparing (5. 7) and (5. 10), we arrive at the required results. The formulas (5. 10) show also some approximate approaches to the construction of the models of regular structures.

Thus, we have proved the –

Theorem. The deformation of an arbitrary regular elastic structure, possessing the property of quasi-periodicity of the displacements, is identical "in the large" with the deformation of the homogeneous anisotropic medium, characterized by the relations (5.7).

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ECHO SIGNAL OF A FINITE SPHERICAL PULSE FROM AN ELASTIC CYLINDRICAL SHELL

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An approximate method of calculating the echo signal of a finite, centrally-symmetric pressure pulse from an infinite elastic cylindrical shell in an infinite ideal compressible fluid is proposed. The shell motion is described by using a linear shell theory of Timoshenko type. The problem is solved by a triple application of integral transformations (in time and the longitudinal coordinate, a Fourier transform, and in the polar angle, a Sommerfeld-Watson transform).

The nonstationary interaction of spherical pressure pulses in a fluid with an elastic cylindrical shell has been studied in [1 – 3], where a Laplace time transformation, a Fourier transformation in the longitudinal coordinate, and either a Fourier series expansion [1, 3] or a Fourier transformation [2] in the polar angle have been used to solve the problem. However, calculation of the rapidly varying components of the Fourier-series solution id difficult because of the slow convergence. Difficulties in inverting the transform appear in the application